California State Polytechnic University, Pomona

Aerospace Engineering Department

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Spacecraft Attitude Dynamics and Control

Spacecraft Stability Final Project

By: Eliot Khachi

To: Professor Navid Nakhjiri

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Executive Summary

Methodology

Problem 1: Derive the equations of motion in terms of Eulerian angles and rates, assuming small angles and small rates and $I_x w_0, I_y w_0, \wedge I_Z w_0 \ll h_S$.

Equations of Motion: $[I]\dot{w}+\dot{h}_s \hat{h}+[w]_x([I]w+h_s\hat{h})=T_c+T_d+T_g$

,where T_c is the control torque, T_d is the external disturbance torque, and T_g is the gravity-gradient torque. The wheel spin-axis is $\hat{h}=\begin{bmatrix}0\\-1\\0\end{bmatrix}$.

a.) The wheel spin is $h_S\hat{h}=\begin{bmatrix}0\\-h_s\\0\end{bmatrix}$, and the angular rate of the spacecraft in the body frame with respect to the inertial frame is $w=w_{bI}=\begin{bmatrix}\dot{\phi}-\psi w_0\\\dot{\theta}-w_0\\\dot{\psi}+\phi w_0\end{bmatrix}$, derived from $w_{bI}=w_{bo}+C_{bo}w_{oI}$, where $w_{bo}=\begin{bmatrix}p\\q\\r\end{bmatrix}\approx\begin{bmatrix}\dot{\phi}\\\dot{\theta}\\\dot{\psi}\end{bmatrix}$ is the angular rate of the spacecraft in the body frame with respect to the orbital frame, w_0 is the angular rate of the spacecraft in the orbital frame with respect to the inertial frame, and $C_{bo}=\begin{bmatrix}1&\psi&-\theta\\-\psi&1&\phi\\\theta&-\phi&1\end{bmatrix}$ is the rotation matrix from the body frame to the orbital frame. Therefore, $\dot{w}=\begin{bmatrix}\ddot{\phi}-\dot{\psi}w_0\\\ddot{\theta}\\\ddot{\psi}+\dot{\phi}w_0\end{bmatrix}$, and the Left-Hand-Side (LHS) of the equation consists of: $[I]\dot{w}=\begin{bmatrix}I_x(\ddot{\phi}-\psi w_0)\\I_y\ddot{\theta}\\I_z(\ddot{\psi}+\dot{\phi}w_0)\end{bmatrix}$, $\dot{h}_s\hat{h}=\begin{bmatrix}0\\-\dot{h}_s\\0\end{bmatrix}$,

$[w]_x=\begin{bmatrix}0&-(\dot{\psi}+\phi w_0)&\dot{\theta}-w_0\\\dot{\psi}+\phi w_0&0&-(\dot{\phi}-\psi w_0)\\-(\dot{\theta}-w_0)&\dot{\phi}-\psi w_0&0\end{bmatrix}$, $[w]_x[I]w=\begin{bmatrix}(I_y-I_z)(\phi w_0^2+\dot{\psi}w_0)\\0\\(I_y-I_x)(\psi w_0^2-\dot{\phi}w_0)\end{bmatrix}$, and

$[w]_x h_s\hat{h}=\begin{bmatrix}h_s(\dot{\psi}+\phi w_0)\\0\\h_s(\psi w_0-\dot{\phi})\end{bmatrix}$. The RHS knowns are the maximum disturbance T_d and

$T_g = \frac{3\mu}{R^5}[R_b]_x [I] R_b$, where $[R_b] = \begin{bmatrix} R\theta \\ -R\phi \\ -R \end{bmatrix}$, so $T_g = 3w_0^2 \begin{bmatrix} (I_z - I_y)\phi \\ (I_z - I_x)\theta \\ 0 \end{bmatrix}$, deriving the following equation of motion:

$$\begin{bmatrix} I_x(\ddot{\phi} - \dot{\psi} w_0) + (I_y - I_z)(\phi w_0^2 + \dot{\psi} w_0) + h_s(\dot{\psi} + \phi w_0) \\ I_y \ddot{\theta} - \dot{h}_s \\ I_z(\ddot{\psi} + \dot{\phi} w_0) + (I_y - I_x)(\psi w_0^2 - \dot{\phi} w_0) + h_s(\psi w_0 - \dot{\phi}) \end{bmatrix} = \begin{bmatrix} T_{cx} + T_{dx} + 3w_0^2 (I_z - I_y)\phi \\ T_{cy} + T_{dy} + 3w_0^2 (I_z - I_x)\theta \\ T_{cz} + T_{dz} \end{bmatrix}$$

b.) Simplifying the previous equation, including $\dot{h}_s$ in T_{cy} and moving the gravity gradient terms onto the left side of the equations, where $k_g = 3(I_x - I_z) w_0^2$:

$$\begin{bmatrix} I_x \ddot{\phi} - (I_x - I_y + I_z)\dot{\psi} w_0 + 4w_0^2 (I_y - I_z)\phi + h_s(\dot{\psi} + \phi w_0) \\ I_y \ddot{\theta} + k_g \theta \\ I_z \ddot{\psi} + (I_x - I_y + I_z)\dot{\phi} w_0 + w_0^2 (I_y - I_x)\psi + h_s(\psi w_0 - \dot{\phi}) \end{bmatrix} = \begin{bmatrix} T_{cx} + T_{dx} \\ T_{cy} + T_{dy} \\ T_{cz} + T_{dz} \end{bmatrix} \quad \textbf{(1)}$$

assuming $I_x w_0, I_y w_0, \wedge I_Z w_0 \ll h_S$ our final equations of motion are:

$$\begin{bmatrix} I_x \ddot{\phi} + h_s(\dot{\psi} + \phi w_0) \\ I_y \ddot{\theta} + k_g \theta \\ I_z \ddot{\psi} + h_s(\psi w_0 - \dot{\phi}) \end{bmatrix} = \begin{bmatrix} T_{cx} + T_{dx} \\ T_{cy} + T_{dy} \\ T_{cz} + T_{dz} \end{bmatrix} \quad \textbf{(2)}$$

c.) A Control System using equation **(1)** was made using Simulink, whose plots were compared with equation **(2)**, which produced plots with unnoticeable difference and therefore aren't shown.

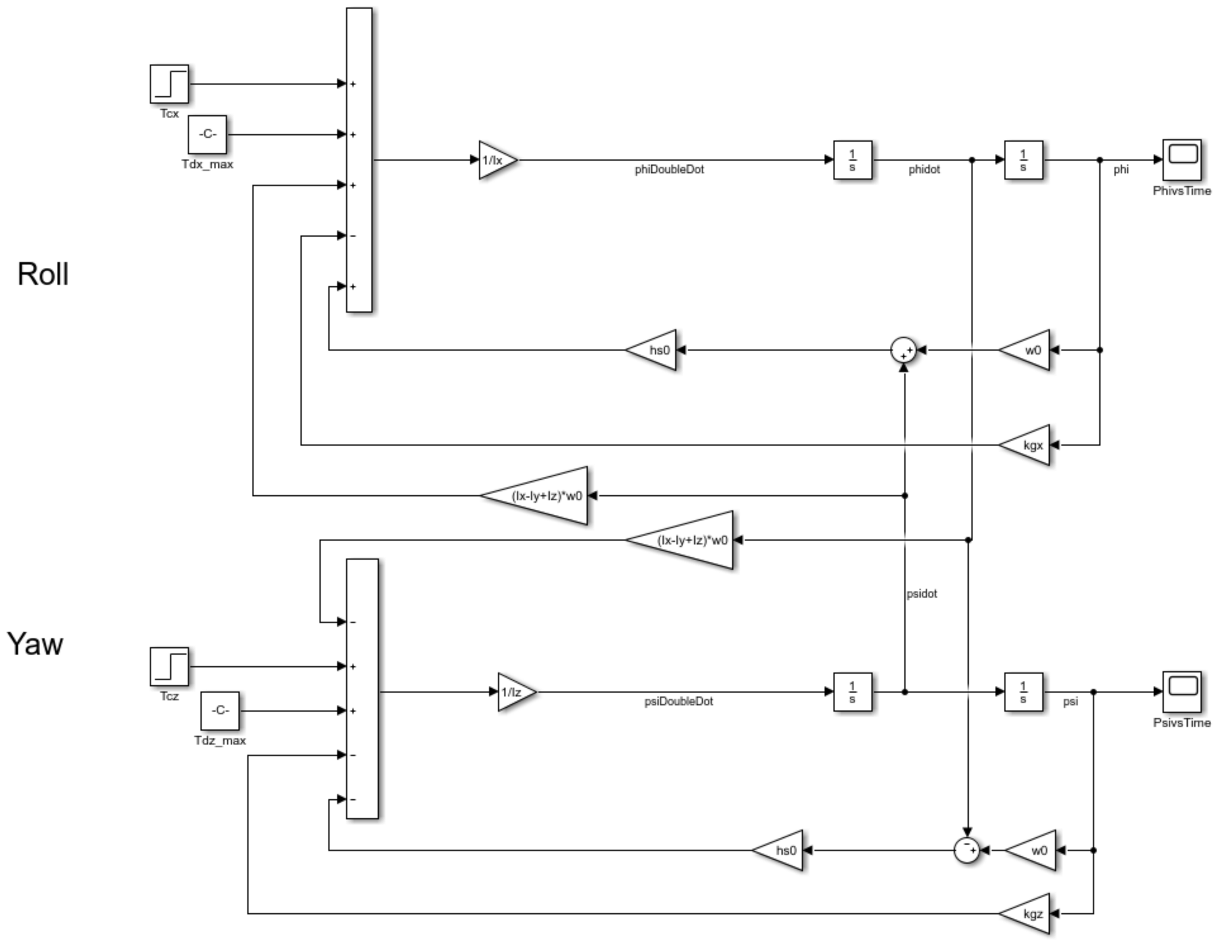

Figure 1.0-1: Coupled Roll/Yaw Simulink Model

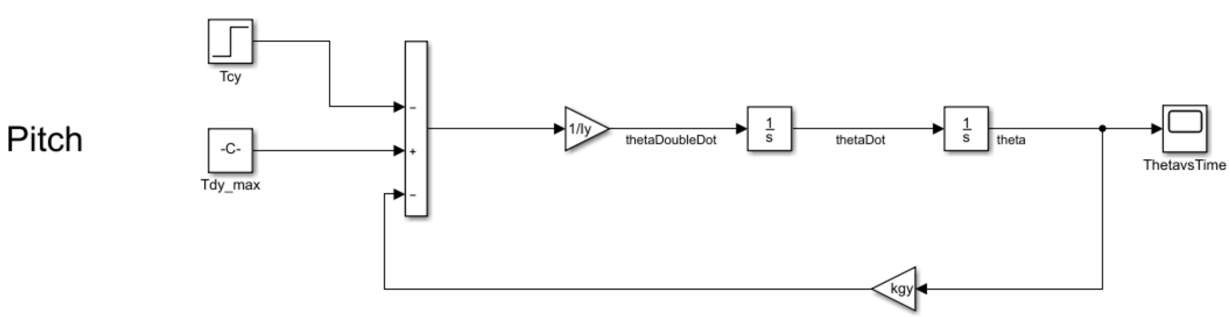

Figure 1.0-2: Uncoupled Pitch Simulink Model

d.) Results from raw Simulink model with 0 control input:

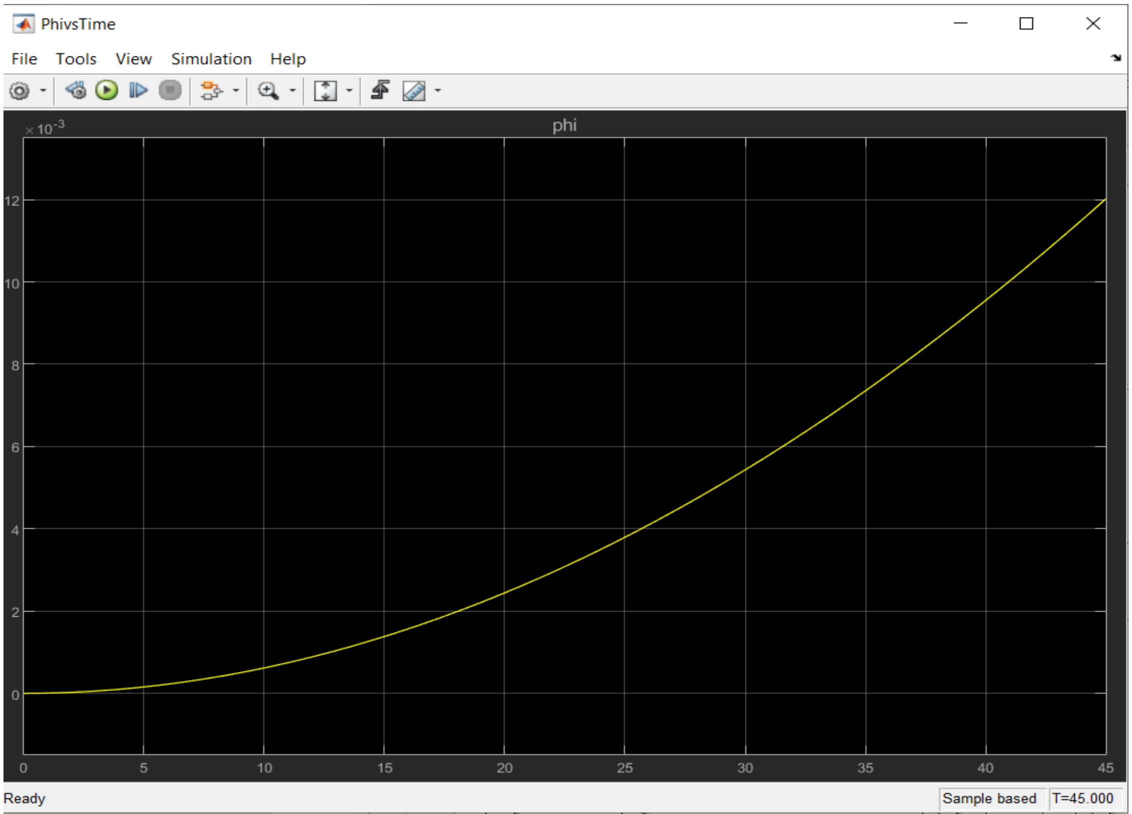

Figure 1.0-3: Roll ϕ (radians) vs Time

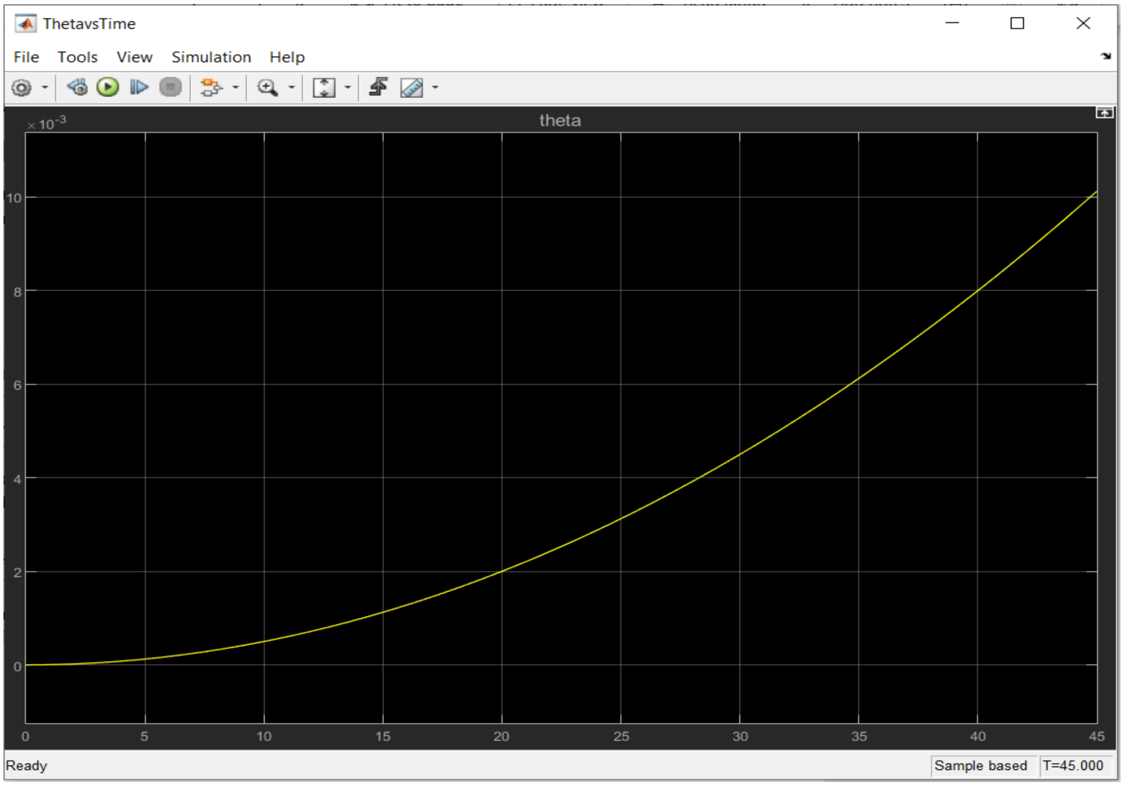

Figure 1.0-4: Pitch θ (radians) vs Time

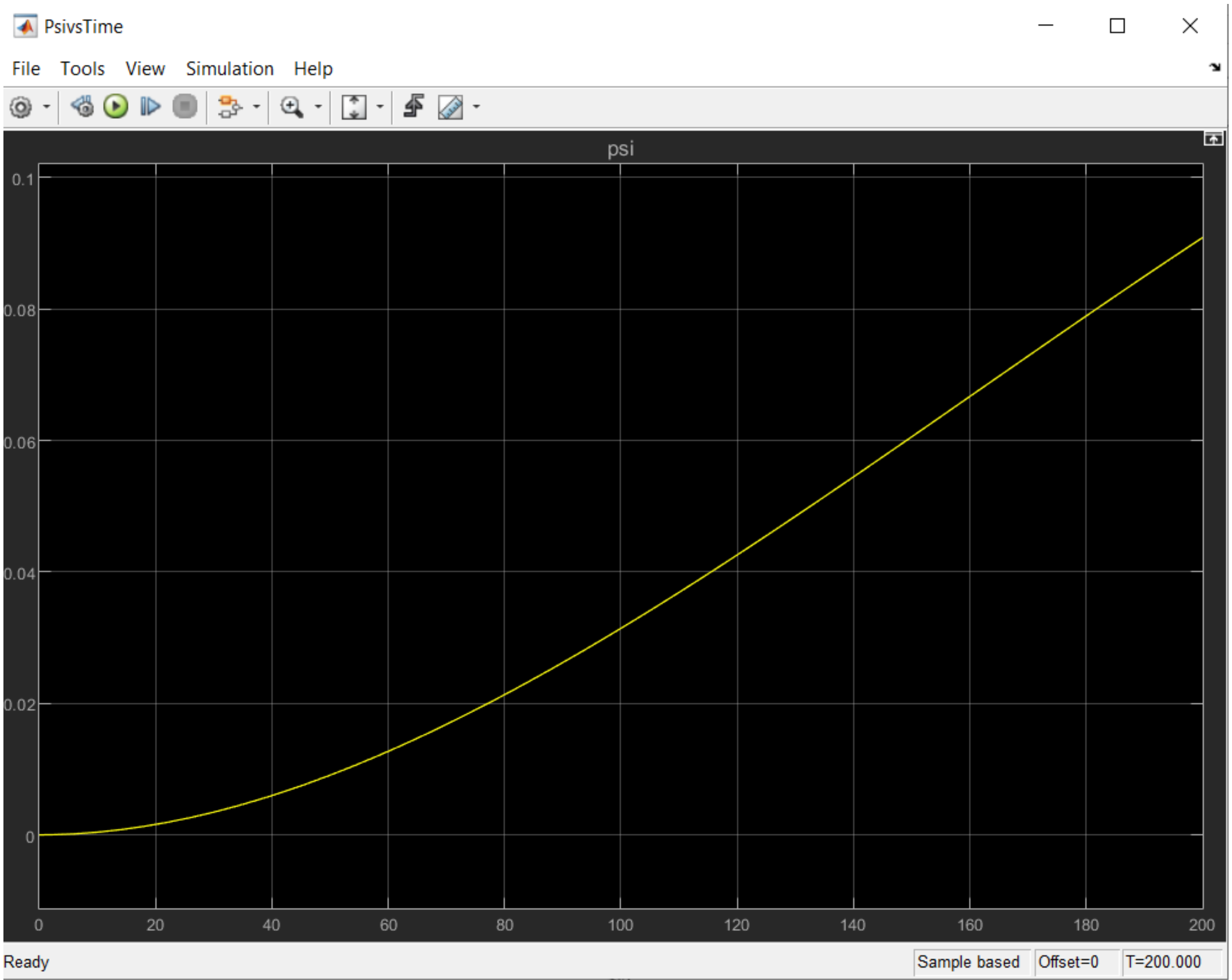

Figure 1.0-5: Yaw ψ (radians) vs Time

Discussion: Derived from the equations of motion, a coupled roll/yaw and an uncoupled pitch Simulink model was created, as show in *Figure 1.0-1* and *Figure 1.0-2*. The results of these control models with 0 control input yields *Figures 1.0-3 to 1.0-5*, which depict the natural instability of the spacecraft given a maximum disturbance of $5\,\mu N \cdot m$. Roll, Pitch and Yaw angles exponentially increase, causing roll to exceed the maximum allowable steady-state error in under 20 seconds, pitch in just under 20 seconds, and yaw in under 3 minutes.

Problem 2: The pitch controller is to be of the modified PD type

$$T_{cy}(t) = k_p^p(\theta_d - \theta) - k_d^p\dot{\theta} \quad \textbf{(3)}$$

a.) The plant transfer function for the pitch angle is $TF = \dfrac{\theta(s)}{T_{cy}(s) + T_{dy}(s)} = \dfrac{1}{I_y s^2 + kg}$ where

This is the form that was used in the Simulink model as seen in *Figure 2.0-1* below.

b.) Implementing the plant transfer function and including the pitch controller and disturbance, we get:

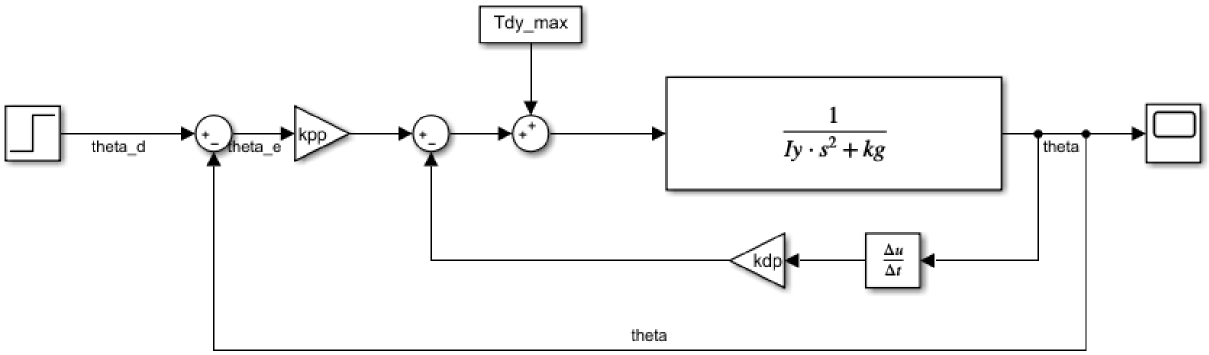

Figure 2.0-1: Controller-Integrated Pitch Angle Simulink Model

c.) The expression for the steady-state pitch error due to a pitch disturbance torque, $T_{dy}(s)=\frac{T_{dy,max}}{s}$, is $\theta_{ss}=\lim_{s\to 0} sE(s)$, where $E(s)=\theta_E(s)=\frac{T_{dy}(s)+T_{cy}(s)+k_d^p s\theta(s)}{k_p^p}=¿$ $\frac{\frac{T_{dy,max}}{s}+T_{cy}(s)+k_d^p s\theta(s)}{k_p^p}$ is derived from equation **(3)**.

Therefore $\theta_{ss}=\lim_{s\to 0} sE(s)=\frac{T_{dy,max}}{k_p^p}=\frac{5*10^{-6}}{0.0015}=0.00333$. Alternatively, we can assume the steady-state error is 0 because we have a type 2 system and step input.

d.) The block diagram in *Figure 2.0-1* was reduced by moving the summing-point of the disturbance torque before the summing-point of the derivative gain, and then obtaining the transfer function for the inner closed-loop, yielding:

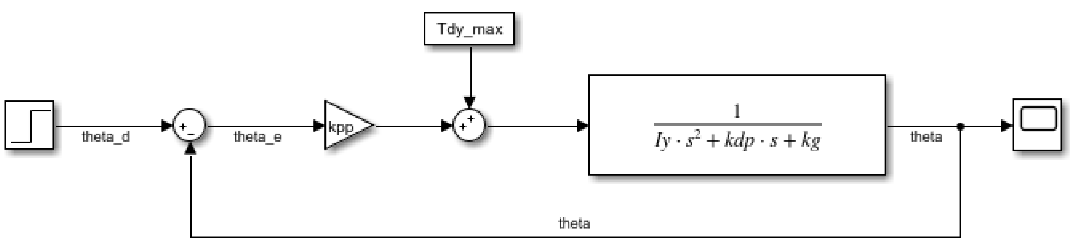

Figure 2.0-2: Reduced Pitch Angle Block Diagram

Therefore the closed-loop transfer function from desired pitch angle to actual, neglecting disturbance, is $\frac{\theta(s)}{\theta_d(s)}=\frac{k_p^p}{I_y s^2+k_d^p s+kg+k_p^p}$

e.) The pitch control gains are $k_p^p = w_n^2 I_y, k_d^p = 2\zeta w_n I_y$,where ζ is solved using closed-loop pitch specification $M_p = \exp\left(\frac{-\pi\xi}{\sqrt{1-\zeta^2}}\right) = 0.30$, and w_n is solved using $t_s = \frac{\ln\left(0.02\sqrt{1-\zeta^2}\right)}{-\zeta w_n} = 200$. Yielding $k_p^p = 0.00155$ and $k_d^p = 0.00199$.

Discussion: The pitch angle is successfully controlled to its desired input with a settling time of 200 seconds and maximum overshoot of 30%. In *Figure 2.0-3* below is a step response to a desired theta value of 0.5 radians.

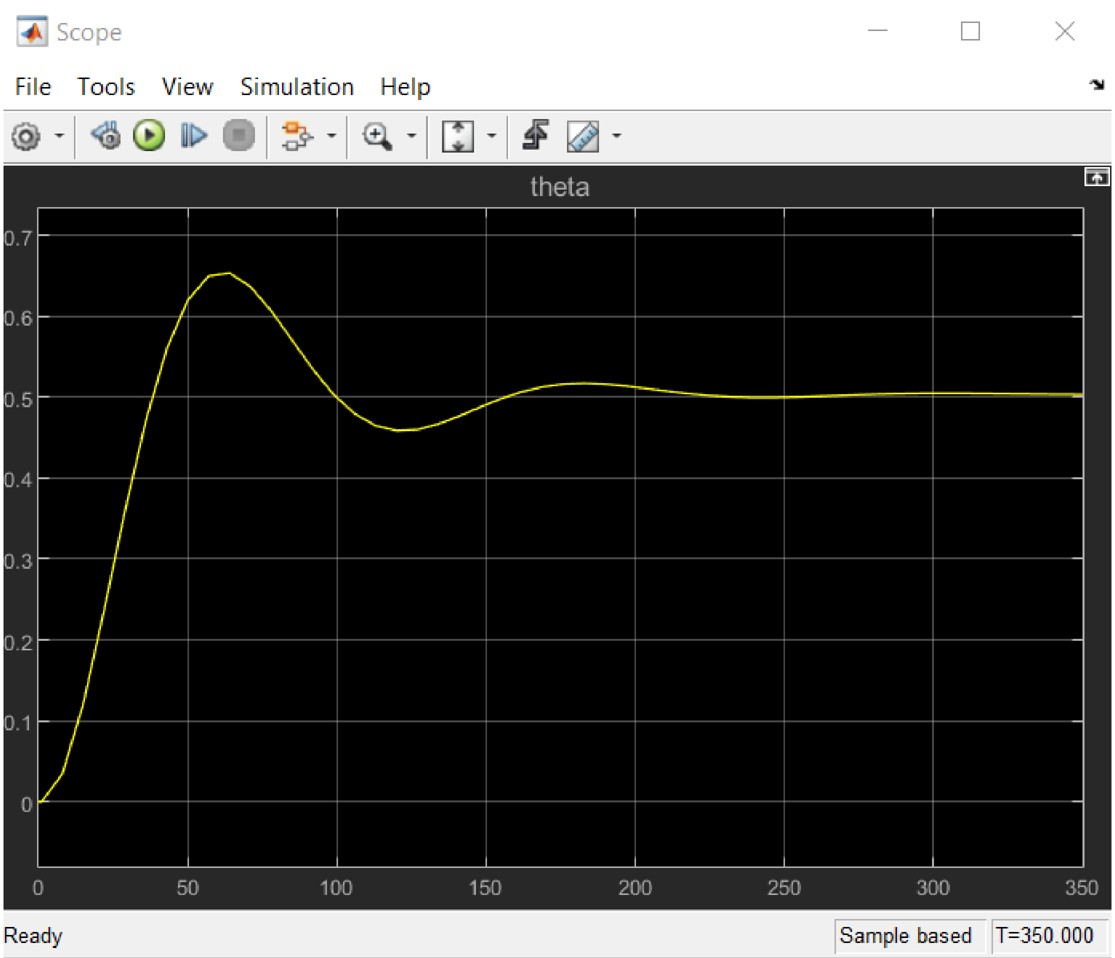

Figure 2.0-3: Pitch Angle Step Response

Problem 3: The control law for the roll/yaw loop are the following

$$T_{cx} = -\left(k_p^{ry}\phi + k_d^{ry}\dot{\phi}\right) \quad \textbf{(4)}$$

$$T_{cz} = \alpha\left(k_p^{ry}\phi + k_d^{ry}\dot{\phi}\right) \quad \textbf{(5)}$$

where $\alpha > 0, k_p^{ry} > 0 \wedge k_p^{ry} > 0$.

a.)

b.) Substituting equation **(4)** and **(5)** into equation **(2)** and taking the Laplace transform gives us the closed-loop roll/yaw equations of motion in Laplace domain in the matrix form:

$$\begin{pmatrix} \phi(s) \\ \psi(s) \end{pmatrix} = \frac{1}{\Delta} ¿,$$

Where

$\Delta = s^4$

$+ s^3 k_d^{ry}$

$+ s^2 ¿$

$+ s^1 ¿$

$+ s^0 ¿$

c.) Therefore for $\alpha > 0, k_p^{ry} > 0 \wedge k_p^{ry} > 0$ the transfer functions for roll and yaw would be:

$$\frac{\phi(s)}{T_{dx}(s)} = \frac{I_x}{\Delta}\left(s^2 + \frac{h_s w_0}{I_z}\right)$$

$$\frac{\phi(s)}{T_{dz}(s)} = \frac{-I_z}{\Delta}\left(\frac{s h_s}{I_x}\right)$$

$$\frac{\psi(s)}{T_{dx}(s)} = \frac{I_x}{\Delta}\left(\frac{s}{I_z}\left(h_s + \alpha k_d^{ry}\right) + \alpha k_p^{ry}\right)$$

$$\frac{\phi(s)}{T_{dz}(s)} = \frac{-I_z}{\Delta}\left(s^2 + \frac{1}{I_x}\left(h_s w_0 + k_p^{ry} + s * k_d^{ry}\right)\right)$$

d.)